

REAL 2-REGULAR CLASSES AND 2-BLOCKS

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ABSTRACT. Suppose that G is a finite group. We show that every 2-block of G has a defect class which is real.

As a partial converse, we show that if G has a real 2-regular class with defect group D and if $\mathbf{N}(D)/D$ has no dihedral subgroup of order 8, then G has a real 2-block with defect group D .

More generally, we show that every 2-block of G which is weakly regular relative to some normal subgroup N has a defect class which is real and contained in N . We give several applications of these results and also investigate some consequences of the existence of non-real 2-blocks.

1. INTRODUCTION

Let G be a finite group and let p be a prime divisor of $|G|$. If B is a p -block of G , then the complex conjugates of the irreducible characters in B form another p -block, B° , of G . We say that B is a real p -block if $B = B^\circ$. In the case that the prime p is 2, it is easy to prove that a real 2-block has at least one real defect class. One of the main results of this paper, Theorem 3.5, shows that in fact an arbitrary 2-block has a real defect class. As a consequence, in order for G to have a 2-block with defect group D , it must have at least one real 2-regular class with defect group D .

The converse of this last statement is false. For instance, the symmetric group S_4 has a real 2-regular class of defect zero, yet it has no 2-blocks of defect zero. However, Theorem 4.8 establishes the following partial converse: Let D be a 2-subgroup of G and let $\mathbf{N}(D)$ denote its normalizer in G . Suppose that $\mathbf{N}(D)/D$ has no dihedral subgroups of order 8. Then G has a 2-block with defect group D if and only if G has a real 2-regular class with defect group D . We conclude Section 4 with a number of examples illustrating the use of this theorem.

In Section 5, we investigate blocks in relation to a normal subgroup N . Our main result is that each 2-block of G which is weakly regular relative to N has a real defect class which is contained in N . If N has odd order, this gives the following purely group-theoretic consequence: N contains a conjugacy class of G with a given defect group if and only if N contains a real conjugacy class of G with the same defect group.

2. PRELIMINARIES

Much of our notation comes from [NT89]. In particular, we fix a p -modular system (K, \mathcal{O}, F) for G and assume that K contains a primitive $|G|$ -th root of unity. (For

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much of this paper, the prime p will be 2, although a number of our results hold for any prime p .) We use \mathcal{P} to denote the unique maximal ideal in \mathcal{O} . If V is an \mathcal{O} -module, and v is an element of V , then v^* will denote the image of v in the F -module V/PV .

If \mathcal{K} is a conjugacy class of G , then \mathcal{K}° will denote the conjugacy class whose members are the inverses of the elements of \mathcal{K} . We say that \mathcal{K} is a *real* class if $\mathcal{K} = \mathcal{K}^\circ$. We use $\mathbf{C}(g)$ to denote the centralizer, and $\mathbf{C}^*(g)$ the extended centralizer, of an element g of G . In particular $\mathbf{C}^*(g)$ is the stabilizer of the set $\{g, g^{-1}\}$ in G . The Sylow p -subgroups of the centralizers of the elements of a conjugacy class \mathcal{K} are called the *defect groups* of \mathcal{K} . When $p = 2$, and \mathcal{K} is a non-identity real class of G , we call the Sylow 2-subgroups of the extended centralizers of the elements of \mathcal{K} the *extended defect groups* of \mathcal{K} .

The elements of G form a basis for the group algebra FG . So every element of FG is of the form $x = \sum_{g \in G} \beta(x, g)g$, where the $\beta(x, g)$ are elements of F . We define the element $x^\circ \in FG$ by $x^\circ = \sum \beta(x, g^{-1})g$. Let S^+ denote the sum of the elements of a subset S of G in FG . The elements \mathcal{K}^+ , where \mathcal{K} is a conjugacy class of G , form a basis for the centre ZFG of FG . If $x \in ZFG$, we use $\beta(x, \mathcal{K}^+)$ to denote the coefficient of \mathcal{K}^+ in x .

The p -blocks of G correspond in a one-to-one manner with the primitive idempotents of ZFG and also with the F -algebra epimorphisms $ZFG \rightarrow F$. If B is a p -block of G with associated primitive idempotent e and F -algebra epimorphism ω , then we will express this association by $B \leftrightarrow e \leftrightarrow \omega$.

Let χ be an irreducible K -character of G . The central character of χ is the map $\omega_\chi : ZOG \rightarrow \mathcal{O}$, given by $\omega_\chi(x) := \chi(x)/\chi(1_G)$, for each $x \in ZOG$. The irreducible K -characters of G are partitioned into blocks, with χ belonging to the block determined by $B \leftrightarrow e \leftrightarrow \omega$ if and only if

$$\omega(\mathcal{K}^+) = \omega_\chi(\mathcal{K}^+)^*,$$

for each conjugacy class \mathcal{K} of G . Set $\omega^\circ(z) := \omega(z^\circ)$, for $z \in ZFG$. Then ω° is an F -algebra epimorphism, and it is clear that $B^\circ \leftrightarrow e^\circ \leftrightarrow \omega^\circ$ is the associated p -block. Moreover, the complex conjugate χ° of χ belongs to B° .

The defect group of a block is defined in Section 3.6 of [NT89]. We collect a number of well-known results about p -blocks in the following lemma.

Lemma 2.1. *Suppose that $B \leftrightarrow e \leftrightarrow \omega$ is a p -block of G , with defect group D . Let \mathcal{K} be a conjugacy class of G . Then $\omega(\mathcal{K}^+) = 0_F$, unless some defect group of \mathcal{K} contains D , and $\beta(e, \mathcal{K}^+) = 0_F$, unless \mathcal{K} is p -regular and some defect group of \mathcal{K} is contained in D .*

We note that, given a p -block $B \leftrightarrow e \leftrightarrow \omega$, there exists at least one class \mathcal{L} such that $\beta(e, \mathcal{L}^+) \neq 0_F$ and $\omega(\mathcal{L}^+) \neq 0_F$. Any such \mathcal{L} is called a *defect class* for B .

In later sections, we will be concerned with showing the existence of real 2-blocks with a given defect group. For this purpose, the proof of Lemma 1.2 of [G88] may be adapted to show the following result.

Lemma 2.2. *Suppose that G has a 2-block with defect group D and let e_D denote the sum of the 2-block idempotents of FG with defect group D . Then G has a real 2-block with defect group D if and only if there exists a real 2-regular class \mathcal{K} with defect group D for which $\beta(e_D, \mathcal{K}^+) \neq 0_F$.*

3. REAL DEFECT CLASSES

Brauer Characters and *Principal Indecomposable Characters* are defined in Section 6.3 of [NT89]. If B is a p -block of G , with defect group D , and if χ is an irreducible K -character in B , then $\chi(1)/|G : D|$ is an element of \mathcal{O} . Moreover, there exists at least one irreducible K -character θ in B for which $\theta(1)/|G : D|$ is a unit in \mathcal{O} . We say that any such θ has *height zero*. The following result is collected from [O66]:

Proposition 3.1. *Suppose that $B \leftrightarrow e \leftrightarrow \omega$ is a p -block of G with defect group D . Let \mathcal{K} be a p -regular class with defect group D . Then*

$$\beta(e, \mathcal{K}^+) = (\dim(B)/|G||\mathcal{K}|)^* \omega(\mathcal{K}^{o+}).$$

Proof. Let θ be an irreducible K -character in B which has height zero. Then $\chi(1)/|\mathcal{K}|$ and $\chi(1)/\theta(1)$ are elements of \mathcal{O} , for each irreducible K -character χ in B . Also $\omega_\chi(\mathcal{K}^+) \equiv \omega_\theta(\mathcal{K}^+) \pmod{\mathcal{P}}$. Thus, given any element k in \mathcal{K} , we have

$$\begin{aligned} \chi(k) &\equiv \frac{\chi(1)}{|\mathcal{K}|} \frac{|\mathcal{K}|}{\theta(1)} \theta(k) \pmod{\mathcal{P}} \\ &\equiv \frac{\chi(1)}{\theta(1)} \theta(k) \pmod{\mathcal{P}} \end{aligned}$$

If ψ is an irreducible Brauer character of G in B then Ψ will denote the corresponding principal indecomposable character. Now ψ is a \mathbb{Z} -linear combination of the restrictions of the ordinary irreducible K -characters in B to the p -regular elements of G . So there exist integers $r_{\chi, \psi}$ such that $\psi = \sum r_{\chi, \psi} \chi$ on p -regular elements, where χ ranges over the irreducible K -characters in B . Thus

$$\psi(k)^* = \left(\sum_{\chi \in B} r_{\chi, \psi} \frac{\chi(1)}{\theta(1)} \right)^* \theta(k)^* = \left(\frac{\psi(1)}{\theta(1)} \right)^* \theta(k)^*.$$

Now $\sum_{\chi \in B} \chi(1)\chi(k^{-1}) = \sum_{\psi \in B} \Psi(1)\psi(k^{-1})$. So

$$\beta(e, \mathcal{K}^+) = \left(|G|^{-1} \sum_{\chi \in B} \chi(1)\chi(k^{-1}) \right)^* = \sum_{\psi \in B} \left(\frac{\Psi(1)}{|G|} \right)^* \psi(k^{-1})^*,$$

taking into account the fact that $\Psi(1)_p \geq |G|_p$ for each Ψ . Thus

$$\beta(e, \mathcal{K}^+) = \left(\sum_{\psi \in B} \frac{\Psi(1)\psi(1)}{|G|\theta(1)} \right)^* \theta(k^{-1})^* = \left(\frac{\dim(B)}{|G|\theta(1)} \right)^* \theta(k^{-1})^*$$

The Proposition now follows from the fact that $\theta(k^{-1})^* = \omega(\mathcal{K}^{o+}) (\theta(1)/|\mathcal{K}|)^*$. \square

Corollary 3.2. *Suppose that $B \leftrightarrow e \leftrightarrow \omega$ is a 2-block of G with defect group D and that \mathcal{K} is a 2-regular class with defect group D . Then $\beta(e, \mathcal{K}^+) = \omega(\mathcal{K}^{o+})$.*

Proof. This follows from Proposition 3.1 and the fact that 1_F is the only non-zero element in the prime field of F . \square

We note in passing that Proposition 3.1 implies the following result of R. Brauer, [B76].

Corollary 3.3. *Suppose that $B \leftrightarrow e \leftrightarrow \omega$ is a p -block of G with defect group D . Set $|G|_p = p^a$ and $|D| = p^d$. Then the p -part of $\dim(B)$ equals p^{2a-d} .*

Proof. It follows from the existence of a defect class (see the note after Lemma 2.1) and Proposition 3.1 that $\dim(B)/|G||G:D|$ is a unit in \mathcal{O} . □

Proposition 3.4. *Suppose that $B \leftrightarrow e \leftrightarrow \omega$ is a 2-block of G with defect group D and let \mathcal{R}_D denote the union of the real 2-regular classes of G which have defect group D . Then $\omega(\mathcal{R}_D^+) = 1_F$ and $\sum_{\mathcal{K} \subseteq \mathcal{R}_D} \beta(e, \mathcal{K}^+) = 1_F$.*

Proof. Let $\mathcal{K}_1, \dots, \mathcal{K}_s$, denote all the real 2-regular classes of G with defect group D and let $\mathcal{L}_1, \mathcal{L}_1^o, \dots, \mathcal{L}_t, \mathcal{L}_t^o$ denote all the non-real 2-regular classes of G with defect group D . Taking into account Lemma 2.1, we have

$$\omega(e) = \sum_{i=1}^s \beta(e, \mathcal{K}_i^+) \omega(\mathcal{K}_i^+) + \sum_{j=1}^t (\beta(e, \mathcal{L}_j^+) \omega(\mathcal{L}_j^+) + \beta(e, \mathcal{L}_j^{o+}) \omega(\mathcal{L}_j^{o+})).$$

Then using Corollary 3.2, we get

$$\begin{aligned} \omega(e) &= \sum_{i=1}^s \omega(\mathcal{K}_i^+)^2 + \sum_{j=1}^t (\omega(\mathcal{L}_j^{o+}) \omega(\mathcal{L}_j^+) + \omega(\mathcal{L}_j^+) \omega(\mathcal{L}_j^{o+})), \\ &= \left(\sum_{i=1}^s \omega(\mathcal{K}_i^+) \right)^2, \quad \text{as } F \text{ has characteristic 2,} \\ &= \omega(\mathcal{R}_D^+)^2. \end{aligned}$$

Thus $1_F = \omega(e) = \omega(\mathcal{R}_D^+)^2$, which in turn implies that $\omega(\mathcal{R}_D^+) = 1_F$. The second equation now follows from Corollary 3.2. □

We now present the main result of this section.

Theorem 3.5. *Every 2-block has a real defect class.*

Proof. Let $B \leftrightarrow e \leftrightarrow \omega$ be a 2-block of G which has defect group D . It follows from the previous theorem that there exists a real 2-regular class \mathcal{K} of G , with defect group D , such that $\omega(\mathcal{K}^+) \neq 0_F$. But $\beta(e, \mathcal{K}^+) \neq 0_F$, using Proposition 3.1. So \mathcal{K} is a real defect class for B . The result follows. □

Our corollary furnishes a necessary condition for the existence of 2-blocks which seems to have been overlooked until now.

Corollary 3.6. *Suppose that G has a 2-block with defect group D . Then G has a real 2-regular conjugacy class with defect group D .*

Example 3.7. Let $n > 1$ be an odd integer and let q be a power of an odd prime p . Suppose that n is relatively prime to $q - 1$. Consideration of rational canonical forms shows that the only real 2-regular class of defect zero in the simple group $\text{SL}_n(q)$ is that containing a regular unipotent element, whose minimal polynomial is $(x - 1)^n$. It is known that $\text{SL}_n(q)$ has 2-blocks of defect 0, so the class of regular unipotent elements is the only real defect class for such blocks. Taking $n = 3$ and $\gcd(3, q - 1) = 1$, we find that the number of 2-blocks of defect 0 is $q(q + 1)/3$. Each

of these blocks contains a unique irreducible character of degree $(q-1)^2(q+1)$, which is not real-valued, and they all have the same real defect class, consisting of elements of order p .

Let $B \leftrightarrow e \leftrightarrow \omega$ be a real 2-block of G with defect group D . Suppose that $\omega(\mathcal{K}^+) \neq 0_F$ and $\omega(\mathcal{L}^+) \neq 0_F$, where \mathcal{K} and \mathcal{L} are real classes of G and \mathcal{K} has defect group D . The first author showed [G88, 2.1] that each extended defect group of \mathcal{K} is contained in some extended defect group of \mathcal{L} , thus refining Lemma 2.1. This need not hold when B is non-real, as the following example shows:

Example 3.8. Let $\mathcal{S}_n(\mathcal{A}_n)$ denote the symmetric (alternating) group of degree n . Suppose that $n = m(m+1)/2$ is a triangular number, where $m > 0$. Then \mathcal{S}_n has a 2-block of defect zero, containing the irreducible character corresponding to the triangular partition $[m, \dots, 3, 2, 1]$. Let χ be an irreducible constituent of the restriction of this character to \mathcal{A}_n . Then χ lies in a 2-block B of defect zero. Now suppose that m is even and greater than 2. The classes of cycle type $[(2m-3)^2, 2m-9, 2m-13, 2m-17, \dots]$, and $[2m-1, 2m-3, 2m-5, \dots, 7, 1^3]$, if $m \equiv 2 \pmod{4}$, of \mathcal{S}_n form single classes \mathcal{K} and \mathcal{L} of \mathcal{A}_n . Moreover both have defect 0, and are real in \mathcal{A}_n , elements being inverted by involutions of cycle type

$$\begin{aligned} [2^{(m/2)^2}], & \quad \text{for } \mathcal{K}, \text{ if } m \equiv 0 \pmod{4}; \\ [2^{(m/2)^2+1}], & \quad \text{for } \mathcal{K}, \text{ if } m \equiv 2 \pmod{4}; \\ [2^{(m/2)^2-1}], & \quad \text{for } \mathcal{L}, \text{ if } m \equiv 2 \pmod{4}. \end{aligned}$$

By repeated applications of the Murnaghan-Nakayama formula, it can be shown that χ takes the value ± 1 on elements of \mathcal{K} or \mathcal{L} . So \mathcal{K} , and \mathcal{L} if $m \equiv 2 \pmod{4}$, are real defect classes for B . However, \mathcal{K} and \mathcal{L} are inverted by non-conjugate involutions.

4. EXISTENCE OF REAL 2-BLOCKS

Given a p -subgroup D of G , we will write $e_D = e_D(G)$ for the sum of the block idempotents of FG with defect group D . In the case that D is the trivial subgroup of G , we will write $e_0 = e_0(G)$ for the sum of the block idempotents of defect 0. Similar notation will be used for subgroups of G . We will also write $\mathcal{R}_0 = \mathcal{R}_0(G)$ for the union of the real p -regular classes which have trivial defect group. Set $W := \mathbf{N}(D)/D$, and let $\pi : F\mathbf{N}(D) \rightarrow FW$ be the F -algebra epimorphism induced by the natural group epimorphism $\pi : \mathbf{N}(D) \rightarrow W$.

Lemma 4.1. *Suppose that k is a p -regular element of G with p -defect group D . Then $\beta(e_D(G), k) = \beta(e_0(W), \pi(k))$.*

Proof. Let $B \leftrightarrow e \leftrightarrow \omega$ be a p -block of G with defect group D . Brauer's First Main Theorem establishes a bijection between the p -blocks of G with defect group D and the p -blocks of $\mathbf{N}(D)$ with defect group D . Let $\tilde{B} \leftrightarrow \tilde{e} \leftrightarrow \tilde{\omega}$ correspond to $B \leftrightarrow e \leftrightarrow \omega$ under this bijection. It follows from Theorem 5.2.15 of [NT89] that

$$(4.2) \quad \beta(e, k) = \beta(\tilde{e}, k).$$

Suppose that k lies in the class \mathcal{K} of $\mathbf{N}(D)$ and that $\pi(k)$ lies in the class \mathcal{L} of W . Then \mathcal{K} is p -regular with p -defect group D . So by Lemma 5.8.9 of [NT89], we have $\pi(\mathcal{K}^+) = \mathcal{L}^+$. It follows that

$$(4.3) \quad \beta(\tilde{e}, \mathcal{K}^+) = \beta(\pi(\tilde{e}), \mathcal{L}^+).$$

Theorem 5.8.7.(ii) of [NT89] implies that $\pi(\tilde{e})$ is a sum of p -blocks of W with trivial defect group. Hence

$$(4.4) \quad \pi(\tilde{e}) = \pi(\tilde{e})e_0(W).$$

Suppose that $B_W \leftrightarrow e_W \leftrightarrow \omega_W$ is a p -block of W , with trivial defect group, such that $\beta(e_W, \mathcal{L}^+) \neq 0_F$. Then $\omega_W \circ \pi : ZF\mathbf{N}(D) \rightarrow F$ is an F -algebra epimorphism. Let $B_N \leftrightarrow e_N \leftrightarrow \omega_N$ be the associated p -block of $\mathbf{N}(D)$. Then

$$\begin{aligned} \omega_N(\mathcal{K}^{o+}) &= \omega_W \circ \pi(\mathcal{K}^{o+}) = \omega_W(\mathcal{L}^{o+}) \\ &= \lambda \cdot \beta(e_W, \mathcal{L}^+), \text{ where } \lambda \neq 0_F, \text{ by Proposition 3.1 and Corollary 3.3} \\ &\neq 0_F, \text{ as } \beta(e_W, \mathcal{L}^+) \neq 0_F. \end{aligned}$$

It follows from Lemma 2.1 that some defect group of B_N is contained in D , the unique p -defect group of \mathcal{K}^o . But D is a normal p -subgroup of $\mathbf{N}(D)$. So every defect group of B_N contains D . We deduce that B_N is a p -block with defect group D . Also $\omega_N(\pi(e_W)) = \omega_W(e_W) = 1_F$. So $e_W = e_W \pi(e_N)$. The result now follows from (4.2), (4.3) and (4.4). \square

We take p to be 2 for the rest of this section and for each positive integer n , we set $\mathcal{Q}_n = \mathcal{Q}_n(G) := \{g \in G \mid g \text{ has order } n\}$.

Lemma 4.5. $e_0 = \mathcal{R}_0^+ + \mathcal{Q}_4^+ + \mathcal{Q}_4^+ \mathcal{Q}_2^+$.

Proof. This follows from the following equalities:

$$\begin{aligned} (1_G + \mathcal{Q}_2^+)^2 &= \mathcal{R}_0^+, \quad \text{by Proposition 4.1 of [M99]} \\ (1_G + \mathcal{Q}_2^+ + \mathcal{Q}_4^+)(1_G + \mathcal{Q}_2^+) &= e_0, \quad \text{by Corollary 5.9 of [M99]}. \end{aligned}$$

\square

If g is a real element of G of defect zero, then it is well known that there exists an involution t which inverts g . So $s := gt$ is also an involution which inverts g . Moreover, if $g = uv$, where u and v are involutions, then both u and v invert g . Any two involutions which invert g are conjugate in $C^*(g)$, and hence come from a single conjugacy class of involutions in G .

We will use D_8 to denote a dihedral group of order 8 and K_4 to denote an elementary abelian group of order 4.

Proposition 4.6. *Suppose that g is a real element of G which has trivial defect group. Let s and t be involutions such that $g = st$. Then*

$$\beta(e_0, g) = 1_F + |\{u \in \mathcal{Q}_2 \mid \langle s, u \rangle \cong D_8, \langle u, t \rangle \cong K_4\}|1_F.$$

Proof. It follows from Lemma 4.5 that

$$\beta(e_0, g) = 1_F + |\Phi_1(g)|1_F,$$

where $\Phi_1(g) := \{(x, y) \in \mathcal{Q}_4 \times \mathcal{Q}_2 \mid xy = g\}$.

Define $(x, y)^t := (x^{-yt}, y^t)$, for $(x, y) \in \Phi_1(g)$. We check that $(x, y)^t \in \Phi_1(g)$ and that $((x, y)^t)^t = (x, y)$. In this way we get an action of the 2-group $\langle t \rangle$ on $\Phi_1(g)$.

Each orbit of $\langle t \rangle$ on $\Phi_1(g)$ has size 1 or 2. So

$$(4.7) \quad \beta(e_0, g) = 1_F + |\Phi_2(g)|1_F,$$

where $\Phi_2(g) := \{(x, y) \in \mathcal{Q}_4 \times \mathcal{Q}_2 \mid xy = g, x = x^{-yt}, y = y^t\}$.

Set $\Phi_3(g) := \{u \in Q_2 \mid \langle s, u \rangle \cong D_8, \langle u, t \rangle \cong K_4\}$. Suppose that $(x, y) \in \Phi_2(g)$ and $u \in \Phi_3(g)$. We check that $yt \in \Phi_3(g)$ and $(su, ut) \in \Phi_2(g)$. Moreover the maps $\Phi_2(g) \rightarrow \Phi_3(g)$, $(x, y) \rightarrow yt$ and $\Phi_3(g) \rightarrow \Phi_2(g)$, $u \rightarrow (su, ut)$ are inverses of each other. Thus $|\Phi_2(g)| = |\Phi_3(g)|$. The proposition now follows from (4.7). \square

In a similar fashion we can show, with the notation above, that

$$\begin{aligned} \beta(e_0, g) &= 1_F + |\{u \in Q_2 \mid \langle s, u \rangle \cong \langle u, t \rangle \cong D_8\}| 1_F \\ &= 1_F + |\{(u, v) \in Q_2 \times Q_2 \mid \langle s, u \rangle \cong \langle u, v \rangle \cong \langle v, t \rangle \cong K_4\}| 1_F. \end{aligned}$$

The next result, which uses Proposition 4.6 in a crucial way, gives a partial converse to Corollary 3.6 and also generalizes Theorem 2 of [T71].

Theorem 4.8. *Suppose that no subgroup of $W := N(D)/D$ is isomorphic to D_8 . Then $\beta(e_D(G), g) = 1_F$, for each real 2-regular element g with defect group D . In particular, the following are equivalent:*

- (a). *G has a real 2-regular element with defect group D ;*
- (b). *G has a 2-block with defect group D ;*
- (c). *G has a real 2-block with defect group D .*

Proof. We have

$$\begin{aligned} \beta(e_D(G), g) &= \beta(e_0(W), \pi(g)), \quad \text{by Lemma 4.1} \\ &= 1_F, \quad \text{using the hypothesis and Proposition 4.6.} \end{aligned}$$

This proves the first assertion.

Suppose that G has a real 2-regular element g with defect group D . Then $\beta(e_D(G), g) = 1_F$. So G has a real 2-block with defect group D , by Lemma 2.2. Thus (a) \implies (c).

The implication (c) \implies (b) is trivial.

The implication (b) \implies (a) follows from Corollary 3.6. This completes the proof. \square

In the situation of Theorem 4.8, we can sometimes guarantee the existence of more than one block with defect group D .

Corollary 4.9. *Suppose that no subgroup of $W := N(D)/D$ is isomorphic to D_8 . Suppose further that G has r real 2-regular classes, $\mathcal{K}_1, \dots, \mathcal{K}_r$, with defect group D , which have non-conjugate extended defect groups. Then G has at least r distinct real 2-blocks with defect group D .*

Proof. It follows from Theorem 4.8 that $\beta(e_D(G), \mathcal{K}_i^+) = 1_F$, for $i = 1, \dots, r$. Corollary 3.2 implies that if $1 \leq i \leq r$, then G has a 2-block $B_i \leftrightarrow e_i \leftrightarrow \omega_i$ with defect group D such that $\omega_i(\mathcal{K}_i^+) \neq 0_F$. Moreover, as in the proof of Lemma 2.2, we may choose each B_i to be real.

As the principal 2-block is the only real 2-block with maximal defect, and the trivial class is the only real class with maximal defect, we may assume that D is not a Sylow 2-subgroup of G . It then follows from Corollary 2.2 of [G88] that $B_i \neq B_j$, for all distinct i, j from $\{1, \dots, r\}$. This completes the proof. \square

Note 4.10. The condition that no subgroup of W is isomorphic to D_8 is equivalent to the condition that all the involutions in a Sylow 2-subgroup of W commute with each other and thus generate an elementary abelian 2-subgroup of W . Corollary

1 of [GL74], taken in conjunction with later work of Bombieri and Thompson on the classification of groups of Ree type, implies that if G is a simple group with no subgroup isomorphic to D_8 , G is isomorphic either to a group of Lie type of rank 1 over a field of even characteristic, or to $\mathrm{PSL}_2(q)$, where $q \equiv 3, 5 \pmod{8}$ and $q > 3$, or to the Janko group J_1 , or to a Ree group ${}^2G_2(3^{2n+1})$. Excluding the group $\mathrm{PSL}_2(2^2)$ which is isomorphic to $\mathrm{PSL}_2(5)$, the groups of Lie type of rank 1 over a field of even characteristic are $\mathrm{PSL}_2(2^n)$, $n \geq 3$, $\mathrm{Sz}(2^{2n+1})$, $n \geq 1$, and $\mathrm{PSU}_3(2^n)$, $n \geq 2$, and each of these groups has a unique 2-block of defect 0, which contains the Steinberg character.

We say that D is a *maximal Sylow 2-intersection* in G if $D = S \cap T$, where $S \neq T$ are Sylow 2-subgroups of G , and if $D \leq P \cap Q$, where $P \neq Q$ are Sylow 2-subgroups of G , then $D = P \cap Q$. The following is a special case of Theorem 4.8.

Corollary 4.11. *Suppose that D is a maximal Sylow 2-intersection in G . Then the conditions (a), (b) and (c) of Theorem 4.8 are equivalent for G .*

Proof. The hypothesis implies that $W = \mathbf{N}(D)/D$ has a trivial intersection Sylow 2-subgroup. An elementary argument (due to M. Suzuki) shows that W has no subgroups isomorphic to D_8 . The result now follows from Theorem 4.8. \square

Our next corollary can also be proved using the Brauer-Suzuki theorem, but the methods developed in this section provide a self-contained approach.

Corollary 4.12. *Suppose that a Sylow 2-subgroup of G is generalized quaternion or cyclic. Then either G has a unique involution or it has a real 2-block of defect 0.*

Proof. Suppose that G has two different involutions s and t . Consider their product st . This is not the identity. We claim that st has odd order. For if this is not the case, there exists an involution u which commutes with both s and t . This contradicts the fact that G contains no elementary abelian subgroup of order 4. Thus st has odd order. Also st is real, since it is inverted by t .

Let S be a Sylow 2-subgroup of the extended centralizer $\mathbf{C}^*(st)$ of st which contains t . Then t is the unique involution in S , but $t \notin \mathbf{C}(st)$. Since $\mathbf{C}(st) \cap S$ is a Sylow 2-subgroup of $\mathbf{C}(st)$, it follows that this latter group has odd order. In particular st has defect zero.

Finally, as a Sylow 2-subgroup contains no dihedral subgroup of order 8, it follows from Theorem 4.8 that G has a real 2-block of defect 0. \square

The Brauer-Suzuki theorem gives the more precise information that any real defect zero element of G lies in $O_{2'}(G)$. This strengthens Proposition 5 of [T74].

Corollary 4.13. *Suppose that G does not possess subgroups H and K with $H \triangleleft K$ and $K/H \cong D_8$. Then all 2-blocks of G have maximal defect if and only if G has a normal Sylow 2-subgroup.*

Proof. Let S be a Sylow 2-subgroup of G . We may suppose that $|S| > 1$. Theorem 4.8 implies that all 2-blocks of G have maximal defect if and only if G has no non-identity real 2-regular elements. Now if S is normal in G , it is elementary to check that G has no non-identity real 2-regular elements. Conversely, suppose that

G has no non-identity real 2-regular elements. Then $\bar{G} := G/O_2(G)$ also has no non-identity real 2-regular elements. We claim that \bar{G} has odd order. For let s and t be involutions in \bar{G} . Then s inverts every element in $\langle st \rangle$ and hence st has 2-power order. It follows from the hypothesis on the subgroups of G that $(st)^2 = 1$. In particular s and t commute. So the involutions in \bar{G} form a normal 2-subgroup. We conclude that \bar{G} has no involutions, which proves our claim. \square

The following result, which generalizes [H68], is a special case of the previous corollary.

Theorem 4.14. *Suppose that a Sylow 2-subgroup S of G is abelian, or is a direct product of a quaternion group of order 8 with an elementary abelian 2-group. Then all 2-blocks of G have maximal defect if and only if S is normal in G .*

The groups S in the theorem above are precisely the Dedekind 2-groups, i.e. those 2-groups all of whose subgroups are normal.

5. BLOCKS AND NORMAL SUBGROUPS

Let N be a normal subgroup of G . We say that a p -block $B \leftrightarrow e \leftrightarrow \omega$ of G covers a p -block $b \leftrightarrow e_b \leftrightarrow \omega_b$ of N if the restriction of some irreducible character in B has an irreducible constituent in b . It can be shown that each block of G covers a G -orbit of blocks of N . More precisely B covers b if and only if $e = e e_b^G$, where e_b^G denotes the sum of the distinct G -conjugates of e_b in G . We will use $\text{Bl}(G|b)$ to denote the set of blocks of G which cover b .

Suppose now that B covers b . We say that B is *weakly regular* (relative to N) if it has maximal defect among the blocks of G which cover b . It follows from Theorem 5.5.16 of [NT89] that the weakly regular blocks in $\text{Bl}(G|b)$ have a common defect group, D say. Moreover, $D \cap N$ is a defect group of b and DN/N is a Sylow p -subgroup of $I(b)/N$, where $I(b)$ is the inertial subgroup of b in G . We call D a defect group of b in G . Those blocks in $\text{Bl}(G|b)$ which are not weakly regular have defect groups strictly contained in D . We now specialize to $p = 2$.

Lemma 5.1. *Suppose that $b \leftrightarrow e_b \leftrightarrow \omega_b$ is a 2-block of N . Then the number of weakly regular blocks which cover b is odd and*

$$\beta(e_b^G, \mathcal{K}^+) = \omega(\mathcal{K}^{o+}),$$

for each 2-block $B \leftrightarrow e \leftrightarrow \omega$ of G which covers b and each 2-regular class \mathcal{K} of G which is contained in N and which has defect group D .

Proof. Let $B_i \leftrightarrow e_i \leftrightarrow \omega_i$, $1 \leq i \leq s$, be a complete list of the blocks of G which cover b , ordered so that B_1, \dots, B_t are all the weakly regular blocks. It follows from Theorem 5.5.5 of [NT89] that

$$(5.2) \quad \omega_i(\mathcal{K}^+) = \omega_j(\mathcal{K}^+),$$

for each class \mathcal{K} of G which is contained in N , and each pair $i, j \in \{1, \dots, s\}$. So we can assume, and we do, that B is weakly regular.

We may write

$$(5.3) \quad e_1 + \dots + e_s = e_b^G = \sum \beta(e_b^G, \mathcal{K}^+) \mathcal{K}^+,$$

where \mathcal{K} ranges over the 2-regular classes of G which are contained in N . Since B covers b , we have

$$(5.4) \quad 1_F = \omega(e) = \omega(e_b^G) = \sum \beta(e_b^G, \mathcal{K}^+) \omega(\mathcal{K}^+).$$

It follows that there is at least one 2-regular class \mathcal{L} contained in N with

$$\beta(e_b^G, \mathcal{L}^+) \neq 0_F \quad \text{and} \quad \omega(\mathcal{L}^+) \neq 0_F.$$

As B has defect group D , we deduce from Lemma 2.1 that D is contained in a defect group of \mathcal{L} . Furthermore, it follows from (5.3) that there is an index j , with $1 \leq j \leq s$, such that $\beta(e_j, \mathcal{L}^+) \neq 0_F$. Since B_j has a defect group contained in D , Lemma 2.1 implies that so too does \mathcal{L} . We conclude that \mathcal{L} has defect group D .

Now B_i has defect group D , for $1 \leq i \leq t$, and B_i has a defect group strictly contained in D , for $t < i \leq s$. So

$$\beta(e_i, \mathcal{L}^+) = \begin{cases} \omega_i(\mathcal{L}^{o+}), & \text{for } 1 \leq i \leq t, \\ 0_F, & \text{for } t < i \leq s, \end{cases} \quad \begin{array}{l} \text{by Proposition 3.1,} \\ \text{by Lemma 2.1.} \end{array}$$

It follows that

$$\beta(e_b^G, \mathcal{L}^+) = \sum_{i=1}^t \omega_i(\mathcal{L}^{o+}) = t^* \omega(\mathcal{L}^{o+}), \quad \text{using (5.3) and (5.2).}$$

In particular, t is odd. Finally, if \mathcal{K} is any 2-regular class of G with defect group D that is contained in N , identical arguments now show that $\beta(e_b^G, \mathcal{K}^+) = \omega(\mathcal{K}^{o+})$. \square

Easy examples show that the following corollary is false when $p \neq 2$.

Corollary 5.5. *Suppose that $b \leftrightarrow e_b \leftrightarrow \omega_b$ is a 2-block of N . Then b is G -conjugate to b^o if and only if some real weakly regular 2-block of G covers b .*

Proof. The ‘if’ part of this statement is straightforward.

Suppose that b is G -conjugate to b^o , and that $B \leftrightarrow e \leftrightarrow \omega$ is a 2-block of G which covers b . Then B covers b^o . So

$$\omega^o(\mathcal{K}^+) = \omega(\mathcal{K}^{o+}) = \omega_b^o(\mathcal{K}^{o+}) = \omega_b(\mathcal{K}^+),$$

for each class \mathcal{K} of G which is contained in N , using Theorem 5.5.5 of [NT89]. It follows that B^o covers b , again using Theorem 5.5.5 of [NT89]. Now B is weakly regular if and only if B^o is weakly regular. So the number of non-real weakly regular blocks in $\text{Bl}(G|b)$ is even. We deduce from Lemma 5.1 that there are an odd number of real weakly regular blocks in $\text{Bl}(G|b)$, proving the ‘only if’ part. \square

We let \mathcal{R}_D^N denote the union of the real 2-regular classes of G which have defect group D and which are contained in N . The following result generalizes Proposition 3.4:

Proposition 5.6. *Suppose that $b \leftrightarrow e_b \leftrightarrow \omega_b$ is a 2-block of N with defect group D in G . Then $\sum_{\mathcal{K} \subseteq \mathcal{R}_D^N} \beta(e_b^G, \mathcal{K}^+) = 1_F$. Also $\omega(\mathcal{R}_D^{N+}) = 1_F$, for each 2-block $B \leftrightarrow e \leftrightarrow \omega$ of G which covers b .*

Proof. We have

$$\begin{aligned} 1_F &= \sum \beta(e_b^G, \mathcal{K}^+) \omega(\mathcal{K}^+), \quad \text{by (5.4),} \\ &= \sum \omega(\mathcal{K}^{o+}) \omega(\mathcal{K}^+), \quad \text{by Lemma 5.1,} \end{aligned}$$

where \mathcal{K} ranges over the 2-regular classes of G which have defect group D and which are contained in N . The proof now proceeds along the same lines as that of Proposition 3.4. \square

We can now prove the following generalization of Theorem 3.5:

Theorem 5.7. *Every 2-block of G which is weakly regular relative to N has a real defect class which is contained in N .*

Proof. This follows immediately from Propositions 3.1 and 5.6. \square

The aim of the remainder of this section is to prove Theorem 5.10. This result, which is concerned with the reality of elements in a non-trivial normal subgroup of odd order, does not appear to have a purely group-theoretic proof. The basic idea for the next lemma goes back to Theorem 5E of [BF55] and was also used in our context by [W77].

Lemma 5.8. *Let N be a normal subgroup of odd order in G . Every conjugacy class of G which is contained in N is a defect class for some real weakly regular 2-block of G .*

Proof. Let \mathcal{K} be a conjugacy class of G which is contained in N , and let k be an element of \mathcal{K} . We define a class function θ of G by

$$\theta := \sum_{\psi} \omega_{\psi}(\mathcal{K}^{o+}) \psi(k) \psi,$$

where ψ ranges over the K -irreducible characters of G . Let P be a Sylow 2-subgroup of G and let g be a non-identity element of P . Since no conjugate of g can be expressed as a product of elements of N , it follows that $\theta(g) = 0$. See, for example, Problem 3.9 of [I94]. Also $\theta(1) = |G|$. So

$$|G : P| = \frac{\theta(1)}{|P|} = (\theta_P, 1_P) = \sum_{\psi} \omega_{\psi}(\mathcal{K}^{o+}) \psi(k) (\psi_P, 1_P),$$

where ψ_P denotes the restriction of ψ to P , and 1_P denotes the principal character of P . Also $\omega_{\psi}(\mathcal{K}^{o+}) \psi(k) = \omega_{\psi^o}(\mathcal{K}^{o+}) \psi^o(k)$ and $(\psi_P, 1_P) = (\psi_P^o, 1_P)$, for each ψ , where ψ^o denotes the complex conjugate of ψ . It follows that

$$|G : P| \equiv \sum \omega_{\psi}(\mathcal{K}^{o+}) \psi(k) (\psi_P, 1_P) \pmod{\mathcal{P}},$$

where ψ ranges over the real K -irreducible characters of G . Since $|G : P|$ is odd, there must be a real-valued irreducible K -character χ with $\chi(k) \not\equiv 0 \pmod{\mathcal{P}}$, $\omega_{\chi}(\mathcal{K}^{o+}) \not\equiv 0 \pmod{\mathcal{P}}$, and $(\chi, 1_P)_P$ an odd integer.

Let $B \leftrightarrow e \leftrightarrow \omega$ be the, necessarily real, 2-block of G which contains χ . Since $\chi(k) \not\equiv 0 \pmod{\mathcal{P}}$ and $\omega_{\chi}(\mathcal{K}^{o+}) \not\equiv 0 \pmod{\mathcal{P}}$, it follows that \mathcal{K} is a defect class for B . So B is weakly regular, using the definition of weak regularity given in Section 5.5 of [NT89]. \square

The following corollary is a straightforward application of a result of P. Fong.

Corollary 5.9. *Suppose that G is solvable. Then G has a 2-block of non-maximal defect if and only if G has a real non-principal 2-block.*

Proof. Any real non-principal 2-block necessarily has non-maximal defect.

On the other hand, suppose that G has a 2-block of non-maximal defect. Then the 2-regular core $O_{2'}(G)$ of G contains a class \mathcal{K} of non-maximal defect, by Theorem (1G)(ii) of [F62]. Thus G has a real 2-block of non-maximal defect, using the previous lemma. \square

We note that Corollary 5.9 need not hold for a group that is not solvable. For example, the Mathieu group M_{11} has exactly three 2-blocks, namely, the principal block and two non-real blocks of defect 0. Theorem 6.6 of this paper shows that this phenomenon can only occur in a group whose Sylow 2-subgroup has order at least 16 (the Sylow 2-subgroup of M_{11} has order 16).

We now prove our main result on the existence of real elements in normal subgroups of odd order.

Theorem 5.10. *Let N be a normal subgroup of G which has odd order, and let D be a 2-subgroup of G . Then G has a conjugacy class with defect group D which is contained in N if and only if G has a real conjugacy class with defect group D which is contained in N .*

Proof. Suppose that G has a conjugacy class which is contained in N and which has defect group D . Then G has a real 2-block B which is weakly regular and which also has defect group D , using Lemma 5.8. So N contains a real defect class for B , by Theorem 5.7. This class has defect group D . The result follows. \square

6. NON-REAL 2-BLOCKS

Let G_D denote the union of the 2-regular classes of G which have defect group D . The following result is similar to Proposition 3.4:

Proposition 6.1. *Suppose that $B \leftrightarrow e \leftrightarrow \omega$ is a 2-block of G with defect group D . Then $\omega(G_D^+) = \begin{cases} 1_F, & \text{if } \omega = \omega^\circ; \\ 0_F, & \text{if } \omega \neq \omega^\circ. \end{cases}$*

Proof. We have

$$\omega(e^\circ) = \sum \beta(e^\circ, \mathcal{K}^+) \omega(\mathcal{K}^+),$$

where \mathcal{K} ranges over the 2-regular classes with defect group D . Using Corollary 3.2, we have

$$\omega(e^\circ) = \sum \omega(\mathcal{K})^2 = \left(\sum \omega(\mathcal{K}^+) \right)^2 = \omega(G_D^+)^2.$$

The proposition follows from the fact that $\omega(e^\circ) = \begin{cases} 1_F, & \text{if } \omega = \omega^\circ; \\ 0_F, & \text{if } \omega \neq \omega^\circ. \end{cases}$ \square

This result can be used to gain additional group-theoretic information when G has a non-real 2-block whose defect group is not a Sylow 2-subgroup of G . Let $G_{2'}$ denote the set of 2-regular elements of G . We begin with the proof of a result that is quite well known.

Lemma 6.2. *Suppose that $B \leftrightarrow e \leftrightarrow \omega$ is a non-principal 2-block of G . Then $\omega(G_{2'}^+) = 0_F$.*

Proof. Let χ be an irreducible K -character in B , and let d be the defect of B . We define a class function θ on G by setting

$$\theta(g) := \begin{cases} 2^d \chi(g), & \text{if } g \in G_{2'}; \\ 0, & \text{if } g \in G \setminus G_{2'}. \end{cases}$$

It is known that θ is an integral combination of the irreducible K -characters in B . See, for example Lemma 3.6.33 (i) of [NT89]. Since B is not the principal block, $(\theta, 1_G) = 0$, where 1_G is the principal character of G . It follows that $\chi(G_{2'}^+) = 0$, and hence that $\omega_\chi(G_{2'}^+) = 0$. The result follows. \square

We proceed to the main result of this section.

Theorem 6.3. *Suppose that $B \leftrightarrow e \leftrightarrow \omega$ is a non-real 2-block of G whose defect group D is not a Sylow 2-subgroup of G . Then there exists a non-identity 2-regular class \mathcal{K} of G , whose defect group strictly contains D , such that $\omega(\mathcal{K}^+) \neq 0_F$.*

Proof. Certainly B is not the principal block of G (as $B \neq B^\circ$). So $\omega(G_{2'}^+) = 0_F$, by the previous lemma. Now $\omega(\mathcal{K}^+) = 0_F$, if a defect group of a class \mathcal{K} does not contain D , using Lemma 2.1. Moreover, $\omega(G_D^+) = 0_F$, by Proposition 6.1, and $\omega(1_G) = 1_F$. Since D is not a Sylow 2-subgroup of G , it follows that $\sum \omega(\mathcal{K}^+) = 1_F$, where \mathcal{K} ranges over the non-identity 2-regular conjugacy classes of G that have a defect group which strictly contains D . The result follows. \square

Theorem 4.8 shows that, in certain circumstances, the existence of a non-real 2-block with defect group D implies the existence of a real 2-block with the same defect group. We end this section by considering an extension of this idea.

The proof of the following result relies on transfer techniques. We omit the details, which are well known.

Lemma 6.4. *Suppose that the finite group G has a dihedral Sylow 2-subgroup. Let t be an involution in G . Then $\mathbf{C}(t)$ has a normal 2-complement.*

While we are primarily interested in the prime 2, the next result, giving a sufficient condition for the existence of a non-principal p -block, holds for any prime p .

Proposition 6.5. *Let D be a p -subgroup of G and let E be a p -subgroup of $\mathbf{N}(D)$ which is not a Sylow p -subgroup of $\mathbf{N}(D)$. Set $W := \mathbf{N}(D)/D$. Suppose that G has a p -regular class with defect group E and that $\mathbf{N}_W(E/D)$ has a normal p -complement. Then G has a non-principal p -block B_G which has a defect group containing E .*

Proof. Recall that $\pi : F\mathbf{N}(D) \rightarrow FW$ is the F -algebra epimorphism induced by the natural group epimorphism $\pi : \mathbf{N}(D) \rightarrow W$. Let k be a p -regular element of G which has defect group E , let \mathcal{K} be the conjugacy class of $\mathbf{N}(D)$ which contains

k , and let \mathcal{L} be the conjugacy class of W which contains $\pi(k)$. Then, as in Lemma 4.1, the class \mathcal{L} has defect group $J := E/D$ in W and $\pi(\mathcal{K}^+) = \mathcal{L}^+$.

Set $H = \mathbf{N}_W(J)$. Now $\mathcal{L} \cap \mathbf{C}_W(J)$ is a p -regular class of H which has defect group J , and by hypothesis H has a normal p -complement. Using Theorem 1. of [T77], we see that H has a p -block $B_H \leftrightarrow \omega_H$, with defect group J , such that $\omega_H((\mathcal{L} \cap \mathbf{C}_W(J))^+) \neq 0_F$. Brauer's first main theorem then implies that W has a p -block $B_W \leftrightarrow \omega_W$, with defect group J , such that $\omega_W(\mathcal{L}^+) \neq 0_F$.

Let $B_N \leftrightarrow \omega_N$ be the unique p -block of $\mathbf{N}(D)$ which dominates B_W , and let R be a defect group of B_N . Then R contains D , by a theorem of Brauer, and J is conjugate in W to a subgroup of R/D , by Theorem 5.8.7 (ii) of [NT89]. Since $E/D = J$, this implies that E is conjugate in $\mathbf{N}(D)$ to a subgroup of R . Also $\omega_N(\mathcal{K}^+) = \omega_W \circ \pi(\mathcal{K}^+) = \omega_W(\mathcal{L}^+) \neq 0_F$. So some conjugate of R is contained in E , the defect group of \mathcal{L} . It follows that B_N has defect group E .

Since $EC(E) \leq \mathbf{N}(D)$, Corollary 5.3.7 of [NT89] guarantees that the induced block $B_G := B_N^G$ is defined. But B_N is not the principal p -block of $\mathbf{N}(D)$, since E is not a Sylow p -subgroup of $\mathbf{N}(D)$. So B_G is not the principal p -block of G , by Brauer's third main theorem (see Theorem 5.6.1 of [NT89]). Finally, it is a standard fact that some defect group of B_G contains E . □

Theorem 6.6. *Suppose that G has a non-real 2-block with defect group D , such that a Sylow 2-subgroup of $W := \mathbf{N}(D)/D$ has order 8. Then G either has a real 2-block with defect group D or it has a non-principal 2-block with defect group strictly containing D .*

Proof. Proposition 4.6 implies G has a real 2-block with defect group D unless possibly W has a dihedral Sylow 2-subgroup. We may therefore suppose that W has a dihedral Sylow 2-subgroup of order 8. Now Theorem 6.3 implies that $\mathbf{N}(D)$ has a non-trivial 2-regular element g with defect group E strictly containing D .

If E is a Sylow 2-subgroup of $\mathbf{N}(D)$, then $\mathbf{N}(D)$ and hence G has a non-principal 2-block with defect group strictly containing D . If E has defect 1 less than maximal, then a standard argument shows that $\mathbf{N}(D)$ has a non-principal 2-block with defect group E , and hence G has a non-principal 2-block with defect group strictly containing D .

If $[E : D] = 2$, then $\mathbf{N}_W(E/D)$ has a normal 2-complement, using Lemma 6.4. So $\mathbf{N}(D)$ has a real non-principal 2-block whose defect group contains E , by Proposition 6.5. It follows that G has a real non-principal 2-block with a defect group which strictly contains D . □

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